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Phugoid Approximation Revisited

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Nomenclature

A = coefficient of the longitudinal characteristic equation
a_0 = coefficient of the longitudinal characteristic equation
a_1 = coefficient of the longitudinal characteristic equation
a_2 = coefficient of the longitudinal characteristic equation
a_3 = coefficient of the longitudinal characteristic equation
B = coefficient of the longitudinal characteristic equation
b_0 = coefficient of the short period equation
b_1 = coefficient of the short period equation
C = coefficient of the longitudinal characteristic equation
c_0 = coefficient of the phugoid equation
c_1 = coefficient of the phugoid equation
D = coefficient of the longitudinal characteristic equation
E = coefficient of the longitudinal characteristic equation
M_q = dimensional variation of pitching moment with pitch rate
M_u = dimensional variation of pitching moment with speed
M_α = dimensional variation of pitching moment with angle of attack
$M_{\dot\alpha}$ = dimensional variation of pitching moment with rate of change of angle of attack
s = Laplace transform variable
U_1 = component of steady state velocity along X
X_q = dimensional variation of X force with pitch rate
X_u = dimensional variation of X force with speed
X_α = dimensional variation of X force with angle of attack
Z_q = dimensional variation of Z force with pitch rate
Z_u = dimensional variation of Z force with speed
Z_α = dimensional variation of Z force with angle of attack
$Z_{\dot\alpha}$ = dimensional variation of Z force with rate of change of angle of attack
ζ_p = damping ratio of the phugoid
ζ_{sp} = damping ratio of the short period
θ = disturbed pitch attitude angle
Θ_1 = steady-state pitch attitude angle
ω_p = frequency of the phugoid
ω_{sp} = frequency of the short period

Introduction

THE longitudinal dynamics of a conventional rigid aircraft consists of the lightly damped, low-frequency phugoid mode representing the long-term translatory motions of the c.g. and the highly damped, high-frequency short-period mode representing rotations about the c.g. Literal approximations were developed many years ago for the short period and the phugoid. The short-period approximation is derived by making use of the fact that the forward speed remains almost invariant during the motion. The perturbed speed and the equation governing its derivative are set to zero in the state-space equations, and the remaining second-order characteristic equation is solved to obtain the approximate equation. The results are excellent. The phugoid is characterized by an almost invariant angle of attack. Likewise, if the perturbed angle of attack and the equation governing its derivative are neglected to obtain the phugoid approximation, the results are disastrous. Instead, proponents of the phugoid approximation neglect the perturbed angle of attack and the pitching moment equation (instead of the equation governing the angle of attack) to derive the approximation. The result is still unsatisfactory.

Great progress has been accomplished concerning the shapes of airplanes, means of computation, and design philosophy. Despite monumental strides made in computational power and the advent of sophisticated mathematical tools on the computer by which stability quartics can be solved before the wink of the eye, the approximate equations are still important because they provide vital information on the functional dependence of the modal parameters on the aerodynamic derivatives, which is invaluable to both the designer and the student alike. The control-system designers of modern-day relaxed stability fighters and transporters require a simple and reasonably accurate reduced-order model. Likewise, the insights provided to the student from an approximate equation cannot be overlooked. Recent publications[1–4] have revived interest in literal approximations.

A new approximation for the phugoid mode is derived in this paper, and it is shown that the result is remarkably accurate by numerical simulation over a wide spectrum database.

Evaluation of Existing Approximations

Following the notation of Roskam,[5] the longitudinal characteristic equation is

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \tag{1}$$

where

$$\begin{aligned}
A &= U_1 - Z_{\dot\alpha}\\
B &= -(U_1 - Z_{\dot\alpha})(X_u + M_q) - Z_\alpha - M_{\dot\alpha}(U_1 + Z_q)\\
C &= X_u[M_q(U_1 - Z_{\dot\alpha}) + Z_\alpha + M_{\dot\alpha}(U_1 + Z_q)]\\
&\quad + M_qZ_\alpha - Z_uX_\alpha + M_{\dot\alpha}g\sin\Theta_1 - M_\alpha(U_1 + Z_q)\\
D &= g\sin\Theta_1(M_\alpha - M_{\dot\alpha}X_u)\\
&\quad + g\cos\Theta_1[Z_uM_{\dot\alpha} + M_u(U_1 - Z_{\dot\alpha})] - M_uX_\alpha(U_1 + Z_q)\\
&\quad + Z_uX_\alpha M_q + X_u[M_\alpha(U_1 + Z_q) - M_qZ_\alpha]\\
E &= g\cos\Theta_1(M_\alpha Z_u - Z_\alpha M_u) + g\sin\Theta_1(M_uX_\alpha - X_uM_\alpha)
\end{aligned} \tag{2}$$

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Table 1 Error in the phugoid approximation

Aircraft	Flight phase	Frequency			Damping		
		Exact	Approximate	% error	Exact	Approximate	% error
A	——	0.1810	0.2082	−15.01	0.1157	0.1062	8.18
B	1	0.2363	0.2702	−14.38	0.0793	0.0993	−25.22
	2	0.1109	0.1307	−17.93	0.1135	0.1023	9.89
	3	0.0948	0.1019	−7.41	0.0693	0.0684	1.32
C	1	0.2984	0.3709	−24.27	−0.0840	0.0000	100.00
	2	0.0666	0.0849	−27.52	−0.0077	0.0000	100.00
	3	0.0733	0.0878	−19.78	−0.0106	0.0000	100.00
D	1	0.2389	0.2678	−12.09	−0.0589	−0.0014	97.65
	2	0.0906	0.0944	−4.12	0.0584	0.0761	−30.31
	3	0.1007	0.1064	−5.71	0.0854	0.0986	−15.36
E	1	0.1682	0.1984	−17.92	−0.1360	0.0000	100.00
	3	0.0269	0.0269	0.05	−0.0741	−0.0537	27.52
F	1	0.1700	0.2082	−22.47	−0.1321	0.0110	108.32
	2	0.0343	0.0362	−5.50	0.1739	0.1888	−8.52
	3	0.0682	0.0749	−9.78	−0.0272	0.0021	107.69

The classical phugoid approximation of Lanchester[6]

$$\omega_p = g\sqrt{2}/U_1 \tag{3}$$

shows that the phugoid frequency is inversely proportional to the forward speed and independent of the aircraft and the flight condition. A more refined estimate[5] is the two-degree-of-freedom approximation:

$$\omega_p = \sqrt{-gZ_u/U_1}, \qquad 2\zeta_p\omega_p = -X_u \tag{4}$$

This is improved in McRuer et al.[7] to obtain the three-degree-of-freedom approximation that captures the effect of the derivative M_u:

$$\omega_p = \sqrt{\frac{g(Z_\alpha M_u - Z_u M_\alpha)}{M_\alpha U_1}}, \qquad 2\zeta_p\omega_p = -X_u + \frac{(X_\alpha - g)M_u}{M_\alpha} \tag{5}$$

Note that this reduces to Eq. (4) when $M_u = 0$. It is reported that all of these equations are inaccurate.

The extent of departure of these approximations from the exact value is first determined. Several aircraft stability and control databases are available. Appendix C of Roskam's test on flight dynamics[5] has been chosen for the numerical studies in this Note. It contains data pertaining to six modern aircraft in a total of 16 flight conditions. The aircraft chosen represent a variety of missions: a small four-place transportation airplane, a 19-passenger commuter airliner, a small jet trainer, a medium-sized high-performance business jet, a supersonic fighter-bomber, and a large wide-body jet transport. The flight conditions range from power approach at sea level to cruise at medium and high altitudes. The database is thus representative of a wide spectrum of airplanes and flight conditions.

Table 1 shows the error in the phugoid approximation [Eq. (5)]. The exact value is determined by using Eq. (2) to solve Eq. (1). The approximation given in McRuer et al.[7] is used to evaluate the phugoid frequency and damping because the two-degree-of-freedom approximation [Eq. (4)] and Lanchester's approximation [Eq. (3)] are its special cases.

The approximation for damping and frequency are both inadequate because of the large errors. Case 2 of aircraft E is not listed because it has two real roots.

The approximate second-order characteristic equation for the short-period mode[5] is

$$s^2 - [M_q + M_{\dot\alpha} + (Z_\alpha/U_1)]s + [(Z_\alpha M_q/U_1) - M_\alpha] = 0 \tag{6}$$

from which the damping and frequency are

$$\omega_{sp} = \sqrt{(Z_\alpha M_q/U_1) - M_\alpha}, \quad 2\zeta_{sp}\omega_{sp} = -[M_q + M_{\dot\alpha} + (Z_\alpha/U_1)] \tag{7}$$

Using the previously mentioned database, it is shown in Table 2 that the preceding expressions are more or less exact. The approximations are seen to be remarkably in agreement with the exact values, which confirms the widespread belief in the short-period equations.

Development of the Approximation

The longitudinal characteristic polynomial [Eq. (1)] is rewritten in the following form:

$$s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0$$

where

$$a_3 = B/A, \qquad a_2 = C/A, \qquad a_1 = D/A, \qquad a_0 = E/A \tag{8}$$

It can be factorized into two quadratics

$$s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = (s^2 + b_1s + b_0)(s^2 + c_1s + c_0)$$

where $b_1 = 2\zeta_{sp}\omega_{sp}$, $b_0 = \omega_{sp}^2$, $c_1 = 2\zeta_p\omega_p$, and $c_0 = \omega_p^2$. If the two quadratics on the right-hand side are multiplied and the coefficients are equated to those of the quartic on the left, the result is

$$a_3 = c_1 + b_1 \tag{9}$$

$$a_2 = c_0 + b_1c_1 + b_0 \tag{10}$$

$$a_1 = b_1c_0 + b_0c_1 \tag{11}$$

$$a_0 = b_0c_0 \tag{12}$$

If the approximate short-period Eq. (6) is used

$$b_1 = -[M_q + M_{\dot\alpha} + (Z_\alpha/U_1)]$$

$$b_0 = (Z_\alpha M_q/U_1) - M_\alpha$$

In Eqs. (9–12), b_1 and b_0 are known and the problem is to determine c_1 and c_0. The solution is nonunique. After a detailed examination of all possibilities, it turns out that the best candidate for c_0 results from Eq. (12): $c_0 = a_0/b_0$. These equations are expanded in terms of the aerodynamic derivatives and simplified. The derivatives Z_q and $Z_{\dot\alpha}$ are neglected in comparison with U_1. It also assumed that $\Theta_1 \approx 0$

$$\omega_p = \sqrt{g(M_uZ_\alpha - M_\alpha Z_u)/(M_\alpha U_1 - Z_\alpha M_q)} \tag{13}$$

The best candidate for c_1 results from Eq. (11): $c_1 = (a_1 - b_1c_0)/b_0$. This is expanded in terms of the aerodynamic derivatives and simplified. The derivatives Z_q and $Z_{\dot\alpha}$ are neglected

Table 2 Error in the short-period approximation

Aircraft	Flight phase	Frequency			Damping		
		Exact	Approximate	% error	Exact	Approximate	% error
A	—	6.0278	6.1013	−1.22	0.6854	0.6829	0.36
B	1	2.8866	2.9245	−1.32	0.6226	0.6164	1.00
	2	6.0846	6.1927	−1.78	0.6786	0.6732	0.79
	3	5.0055	5.0457	−0.80	0.4857	0.4836	0.43
C	1	1.6003	1.6116	−0.71	0.7380	0.7271	1.48
	2	3.4301	3.4577	−0.80	0.8151	0.8145	0.08
	3	2.5707	2.5859	−0.59	0.7238	0.7229	0.12
D	1	1.6019	1.6157	−0.23	0.5666	0.5585	1.41
	2	2.8223	2.8286	−0.22	0.3524	0.3518	0.16
	3	2.9465	2.9562	−0.33	0.3997	0.3991	0.14
E	1	0.7783	0.7603	2.31	0.5995	0.5837	2.65
	3	4.8636	4.8638	−0.01	0.0639	0.0641	−0.30
F	1	0.7734	0.7766	−0.42	0.6174	0.5987	3.03
	2	1.3232	1.3336	−0.78	0.3551	0.3532	0.53
	3	1.2420	1.2598	−1.44	0.4687	0.4654	0.71

Table 3 Accuracy of the new phugoid approximation

Aircraft	Flight phase	Frequency			Damping		
		Exact	Approximate	% error	Exact	Approximate	% error
A	—	0.1810	0.1796	0.76	0.1157	0.1170	−1.16
B	1	0.2363	0.2342	0.88	0.0793	0.0820	−3.36
	2	0.1109	0.1095	1.22	0.1135	0.1155	−1.67
	3	0.0948	0.0943	0.59	0.0693	0.0699	−0.82
C	1	0.2984	0.2981	0.11	−0.0840	−0.0809	3.66
	2	0.0666	0.0663	0.47	−0.0077	−0.0076	2.25
	3	0.0733	0.0731	0.36	−0.0106	−0.0105	1.72
D	1	0.2389	0.2389	0.02	−0.0589	−0.0575	2.39
	2	0.0906	0.0907	−0.03	0.0584	0.0584	0.10
	3	0.1007	0.1006	0.13	0.0854	0.0854	0.01
E	1	0.1682	0.1740	−3.45	−0.1360	−0.1399	−2.89
	3	0.0269	0.0268	0.23	−0.0741	−0.0742	−0.18
F	1	0.1700	0.1730	−1.77	−0.1321	−0.1302	1.42
	2	0.0343	0.0346	−0.65	0.1739	0.1736	0.18
	3	0.0682	0.0678	0.55	−0.0272	−0.0266	2.20

in comparison with U_1. If $\Theta_1 \approx 0$, as was done for frequency, it turns out that in one of the cases, the error in damping becomes as high as 48%. The approximation for damping requires $\sin \Theta_1$ to be retained as it is:

$$2\zeta_p \omega_p = \frac{1}{M_\alpha U_1 - M_q Z_\alpha}\Bigg(-g \sin \Theta_1 M_\alpha + X_u(g \sin \Theta_1 M_{\dot{\alpha}} - M_\alpha U_1 + M_q Z_\alpha) + Z_u\left\{-gM_{\dot{\alpha}} - M_q X_\alpha + \frac{gM_\alpha[U_1(M_{\dot{\alpha}} + M_q) + Z_\alpha]}{M_\alpha U_1 - M_q Z_\alpha}\right\} + M_u\left\{U_1(X_\alpha - g) - \frac{gZ_\alpha[U_1(M_{\dot{\alpha}} + M_q) + Z_\alpha]}{M_\alpha U_1 - M_q Z_\alpha}\right\}\Bigg) \tag{14}$$

Table 3 is a study of its accuracy. In the 15 cases considered, the maximum error in frequency is 3.45% (in 12 of them, the error is less than 1%), whereas in the traditional approximation, the maximum error is 27.5%. The maximum error in damping is 3.66% in the 15 cases considered.

The expression for frequency is a modification of Eq. (5), with the extra term $(-Z_\alpha M_q)$ in the denominator. It is elegant and concise. The expression for damping, although extremely accurate, is rather large, but this is inevitable because the dropping of any term leads to a large inaccuracy in damping.

Conclusions

The phugoid approximation for a conventional aircraft is inaccurate, whereas that of the short period is a fairly true representation. An approximate expression is obtained for the phugoid mode in this paper. It is obtained by writing the fourth-order longitudinal characteristic equation as the product of two quadratics, representing the short period and the phugoid modes. The quadratics are multiplied and the coefficients are equated to the quartic to obtain a new approximation for the phugoid. The results are seen to be excellent.

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